

# Connecting enactive/iconic/symbolic models to mathematical problem types for addition and subtraction

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Dr. Jonathan Brendefur's research has greatly influenced our teaching strategies used in class. He has given permission to share the following information.

# Word Problem Types

and

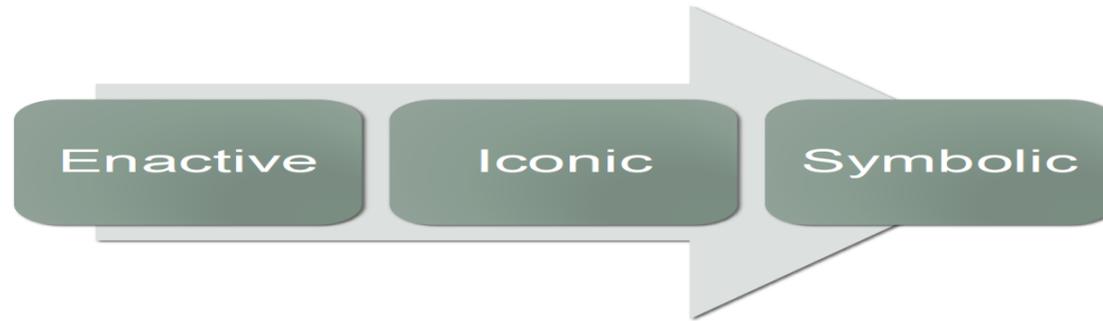
# Common Underlying Structures

JOINING Problems		
Join: Result Unknown JRU	Join: Change Unknown JCU	Join: Start Unknown JSU
Elise had 5 strawberries. Max gave her 8 more strawberries. How many strawberries does Elise now have?	Elise had 5 strawberries. Max gave her some more strawberries. Now Elise has 18 strawberries. How many strawberries did Max give Elise?	Elise had some strawberries. Max gave her 8 more strawberries. Now Elise has 18 strawberries. How many strawberries did Elise have before Max gave her any?
SEPARATING Problems		
Separate: Result Unknown SRU	Separate: Change Unknown SCU	Separate: Start Unknown SSU
Elise had 13 strawberries. She gave 5 strawberries to Max. How many strawberries does Elise have now?	Elise had 13 strawberries. She gave some to Max. Now Elise has 5 strawberries. How many strawberries to Elise give Max?	Elise had some strawberries. She gave 5 to Max. Now Elise has 8 strawberries. How many strawberries did Elise have before she gave any to Max?
Part-Part-Whole Problems		
Part-Part-Whole: Whole Unknown (PPW:WU)	Part-Part-Whole: Part Unknown (PPW:PU)	Combination
Elise had 5 big strawberries and 8 small strawberries. How many strawberries does Elise have?	Elise has 13 strawberries. 5 are big and the rest are small. How many are small.	There are 13 students on the playground. Some are boys and some are girls. How many could be boys and how many could be girls?
Compare Problems		
Compare Difference Unknown (CDU)	Compare Quantity Unknown (CQU)	Compare Referent Unknown (CDU)
Elise has 8 strawberries. Max has 5 strawberries. How many more strawberries does Elise have than Max?	Max has 5 strawberries. Elise has 3 more strawberries than Max. How many strawberries does Elise have?	Elise has 8 strawberries. She has 3 more strawberries than Max. How many strawberries does Max have?
Multiplication & Division Problems		
Multiplication	Partitive Division	Measurement Division
Elise has 4 piles of strawberries. There are 3 strawberries in each pile. How many strawberries does Elise have?	Elise has 12 strawberries. She wants to give them to 3 children. If Elise gives the same amount to each child, how many strawberries will each child get?	Elise had some strawberries. She gave them to some children. Elise gave each child 3 strawberries. How many children were given strawberries?

# Concerns with Key Word Strategy

- **Key words are often misleading, suggesting an operation that is incorrect**
- Many problems have no key words
- Key words do not work in two-step word problems

# Visual Representation: Enactive/Iconic/Symbolic Modeling



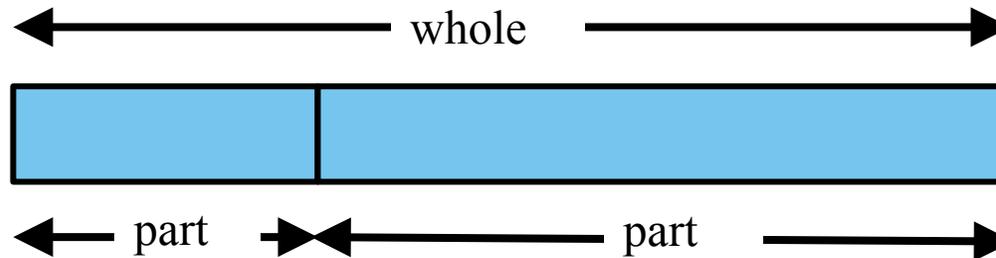
- Modeling is primarily seen as an organizing activity in which situations are **structured in terms of mathematical relationships** *Dlamini, E. (2014), p.4.*
- Transition from being a model “of” a situation to becoming a model “for” mathematical reasoning

# Objectives

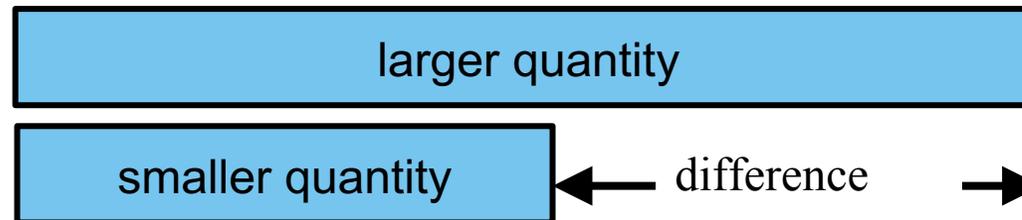
- Use bar models to represent the underlying structure of a problem type.
- Flexibly solve problems with enactive, iconic, and symbolic representation.
- Use reasoning to develop situations that reflect particular problem types.
- Develop understanding of the relationship between addition and subtraction.

# Types of bar models: *Addition and Subtraction*

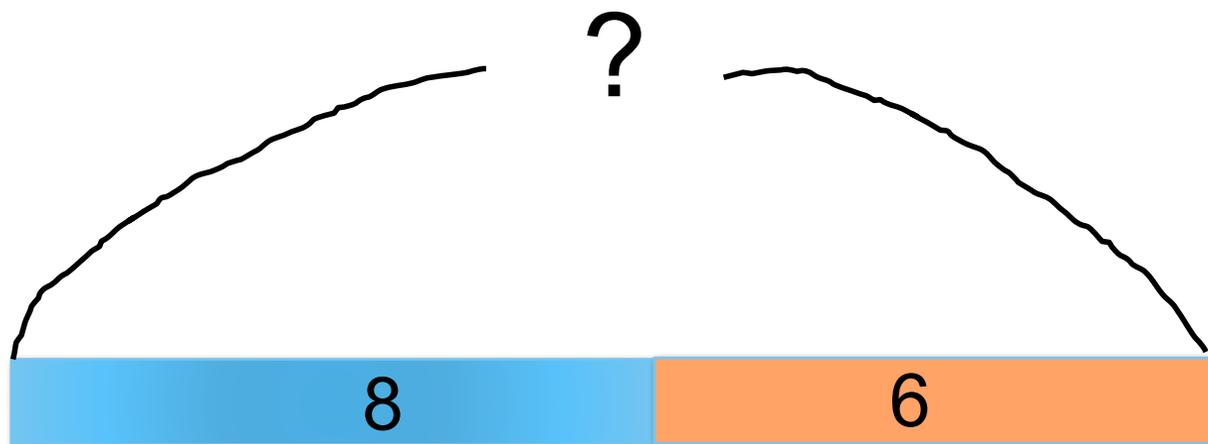
❖ Part-whole



❖ Comparison



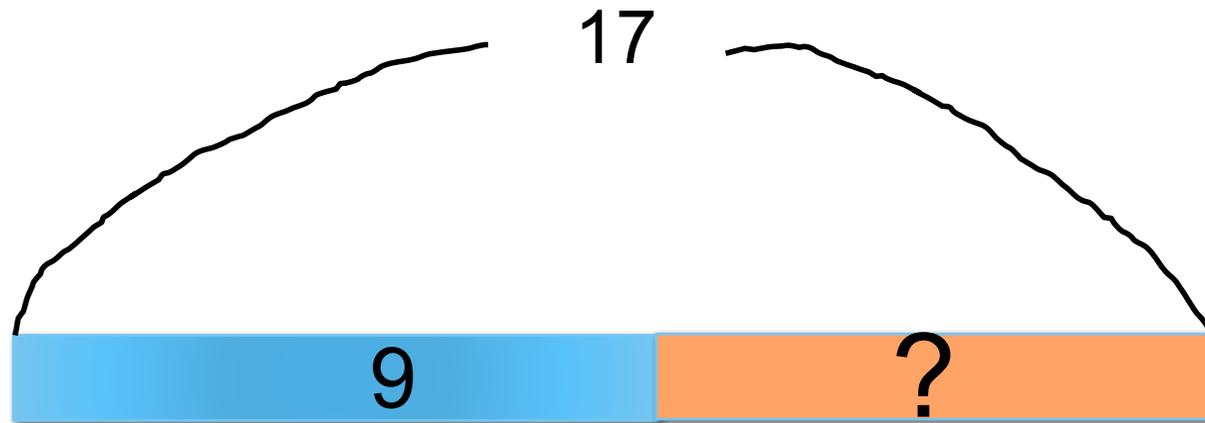
Jack has 8 cars. His mom gave him 6 cars. How many toy cars does Jack have?



$$8 + 6 =$$

Structure	
Total	Join <i>Problem Type</i>
	Join <b>Result Unknown</b>

Bella read nine pages at home. She read some more at school. Bella has now read seventeen pages. How many pages did she read at school?

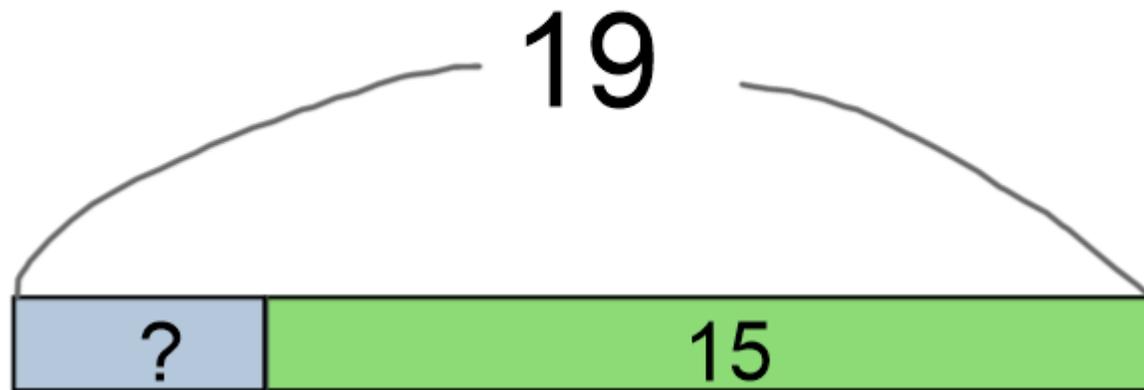


Structure	
Change	Join <i>Problem Type</i>
Join Change Unknown	

$$9 + ? = 17$$

$$17 - 9 = ?$$

Owen had some crayons. Kendall gave him 15 more crayons. Now Owen has 19 crayons. How many crayons did Owen have to start with?



$$? + 15 = 19$$

$$19 - 15 =$$

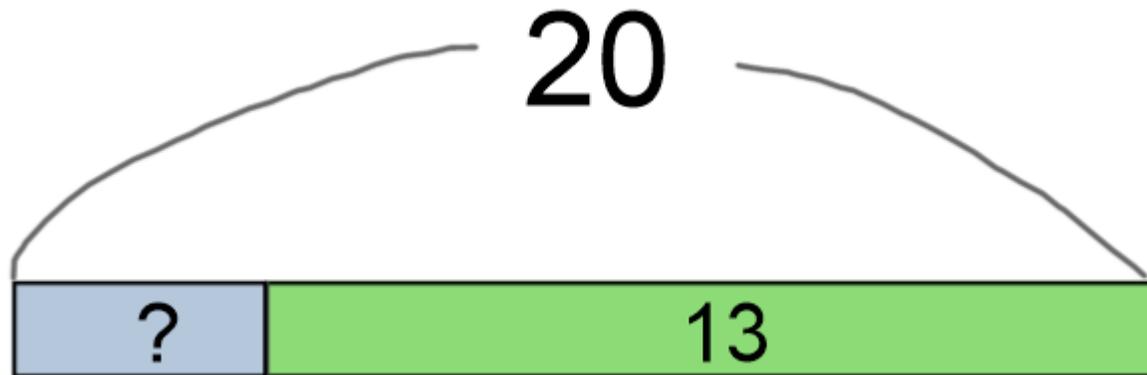
## Structure

Change

Join *Problem Type*

Join **Start  
Unknown**

There were 20 students in the cafeteria, but 13 students left.  
How many students were still in the cafeteria?



$$20 - 13 = ?$$

$$13 + ? = 20$$

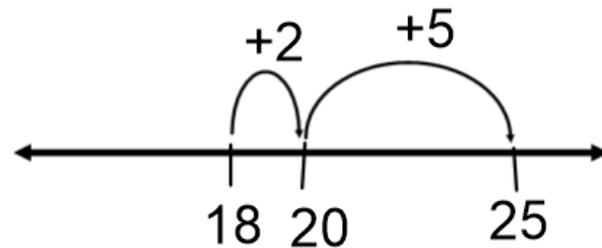
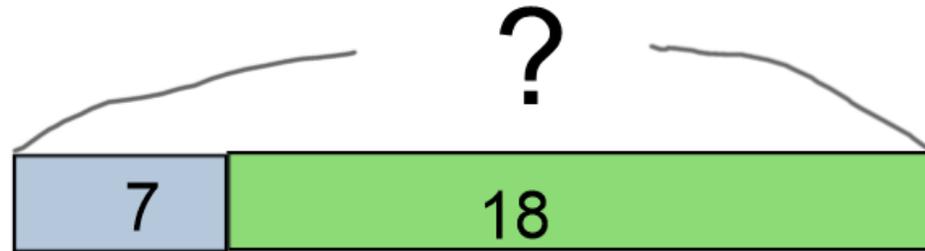
## Structure

Change

Separate *Problem Type*

Separate **Result Unknown**

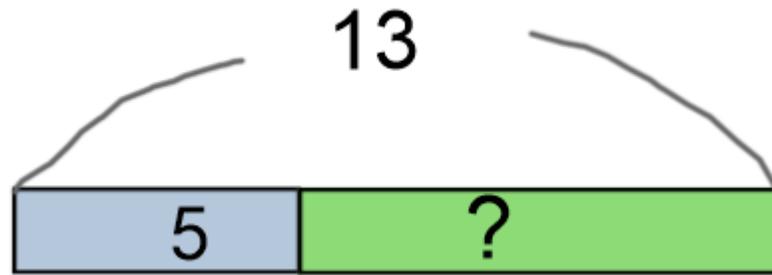
Jay had some chips. He gave Caleb seven chips. Now Jay has eighteen potato chips. How many chips did Jay have before he gave some chips to Caleb?



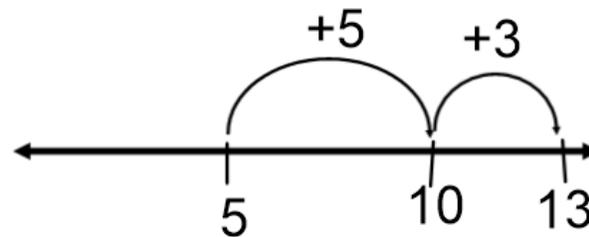
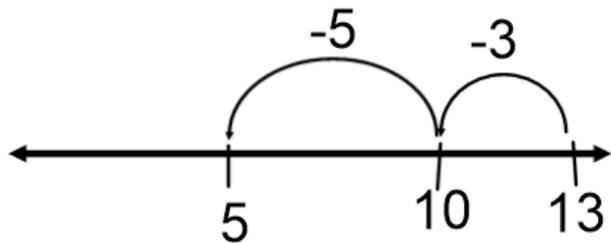
Structure	
Total	Separate <i>Problem Type</i>
	Separate <b>Start Unknown</b>

$$\begin{aligned} ? - 7 &= 18 \\ 7 + 18 &= ? \end{aligned}$$

Lisl had 13 marbles. She gave some to Thomas. Now she has 5 marbles left. How many marbles did Lisl give to Thomas?



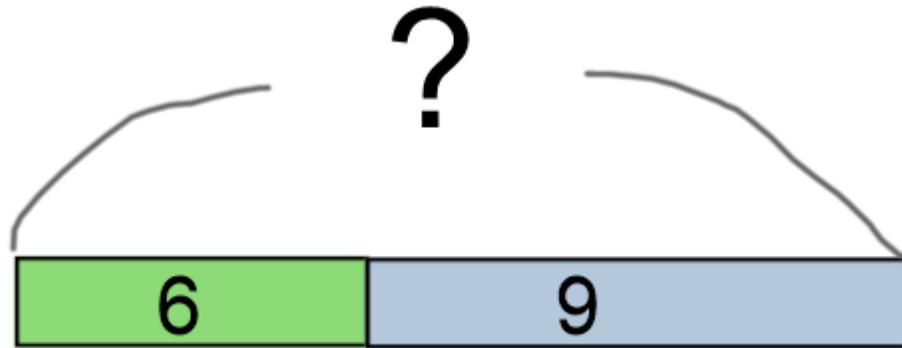
Structure	
Change	Separate <i>Problem Type</i>
	Separate <b>Change</b> <b>Unknown</b>



$$5 + ? = 13$$

$$13 - 5 = ?$$

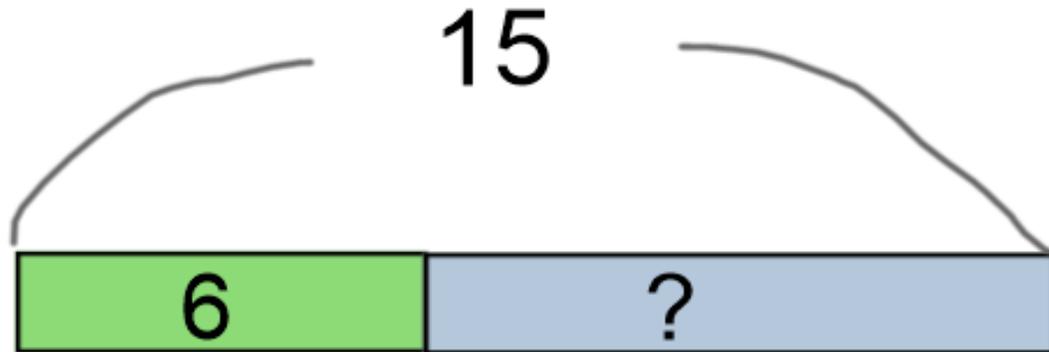
Jack has six green toy cars and nine blue toy cars. How many toy cars does Jack have?



$$6 + 9 = ?$$

Structure	
Total	Part-Part- Whole <i>Problem Type</i>
	PPW: <b>Whole Unknown</b>

Jack has fifteen toy cars. He has six green toy cars and the rest are blue. How many blue toy cars does Jack have?



$$15 - 6 = ?$$
$$6 + ? = 15$$

Structure	
Part-Part-Whole*	Part-Part-Whole <i>Problem Type</i>
	PPW: <b>Part Unknown</b>

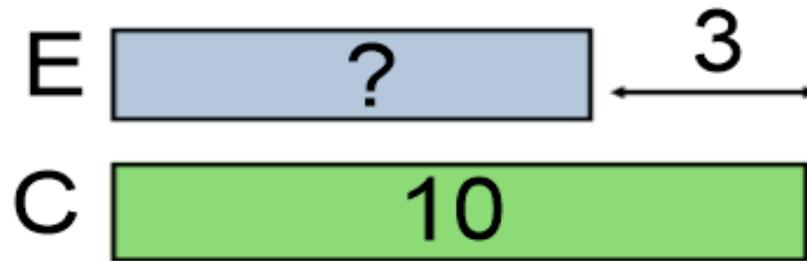
Justin's shoe is 11 inches long. Annie's shoe is 7 inches long. How much longer is Justin's shoe than Annie's shoe?



$$11 - 7 = ?$$
$$7 + ? = 11$$

Structure	
Compare	Compare <i>Problem Type</i>
	CDU: <b>Difference Unknown</b>

Carter has 10 cents. He has three more cents than Ellie.  
How much money does Ellie have?

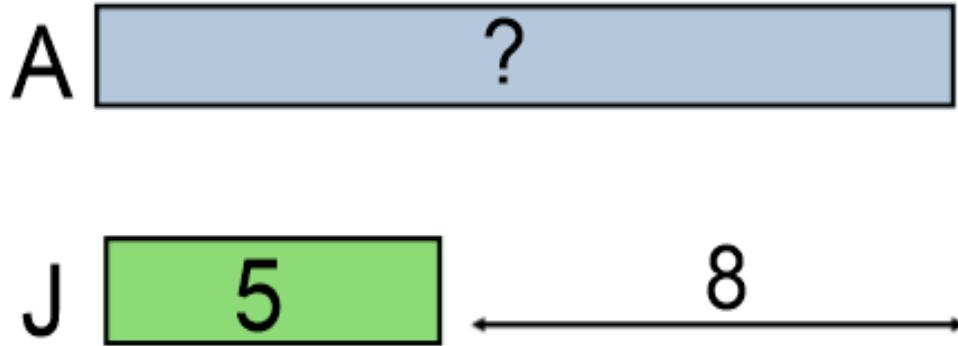


$$10 - 3 = ?$$

$$? + 3 = 10$$

Structure	
Compare	Compare <i>Problem Type</i>
CRU: Referent Unknown	

Jon has 5 marbles. Annie has 8 more than Jon. How many marbles does Kate have?



Structure	
Compare	Compare <i>Problem Type</i>
CQU: <b>Quantity Unknown</b>	

$$5 + 8 = ?$$

$$? - 5 = 8$$

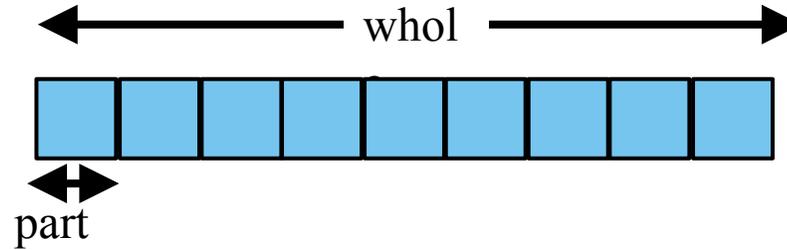
# Multiplication and Division

# Word Problems **Types** and Common Underlying **Structures**: *Multiplication, and Division*

	Unknown Product	Group Size Unknown ("How many in each group?" Division)	Number of Groups Unknown ("How many groups?" Division)
	$3 \times 6 = ?$	$3 \times ? = 18$ , and $18 \div 3 = ?$	$? \times 6 = 18$ , and $18 \div 6 = ?$
Equal Groups	<p>There are 3 bags with 6 plums in each bag. How many plums are there in all?</p> <p><i>Measurement example.</i> You need 3 lengths of string, each 6 inches long. How much string will you need altogether?</p>	<p>If 18 plums are shared equally into 3 bags, then how many plums will be in each bag?</p> <p><i>Measurement example.</i> You have 18 inches of string, which you will cut into 3 equal pieces. How long will each piece of string be?</p>	<p>If 18 plums are to be packed 6 to a bag, then how many bags are needed?</p> <p><i>Measurement example.</i> You have 18 inches of string, which you will cut into pieces that are 6 inches long. How many pieces of string will you have?</p>
Arrays, <sup>4</sup> Area <sup>5</sup>	<p>There are 3 rows of apples with 6 apples in each row. How many apples are there?</p> <p><i>Area example.</i> What is the area of a 3 cm by 6 cm rectangle?</p>	<p>If 18 apples are arranged into 3 equal rows, how many apples will be in each row?</p> <p><i>Area example.</i> A rectangle has area 18 square centimeters. If one side is 3 cm long, how long is a side next to it?</p>	<p>If 18 apples are arranged into equal rows of 6 apples, how many rows will there be?</p> <p><i>Area example.</i> A rectangle has area 18 square centimeters. If one side is 6 cm long, how long is a side next to it?</p>
Compare	<p>A blue hat costs \$6. A red hat costs 3 times as much as the blue hat. How much does the red hat cost?</p> <p><i>Measurement example.</i> A rubber band is 6 cm long. How long will the rubber band be when it is stretched to be 3 times as long?</p>	<p>A red hat costs \$18 and that is 3 times as much as a blue hat costs. How much does a blue hat cost?</p> <p><i>Measurement example.</i> A rubber band is stretched to be 18 cm long and that is 3 times as long as it was at first. How long was the rubber band at first?</p>	<p>A red hat costs \$18 and a blue hat costs \$6. How many times as much does the red hat cost as the blue hat?</p> <p><i>Measurement example.</i> A rubber band was 6 cm long at first. Now it is stretched to be 18 cm long. How many times as long is the rubber band now as it was at first?</p>
General	$a \times b = ?$	$a \times ? = p$ , and $p \div a = ?$	$? \times b = p$ , and $p \div b = ?$

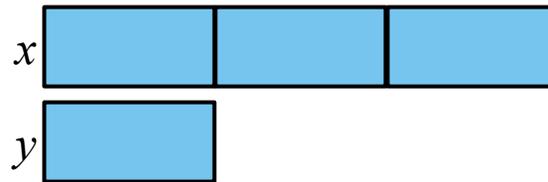
# Types of bar models: *Multiplication and Division*

❖ Part-whole

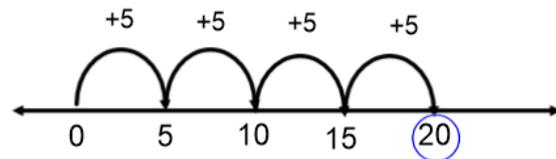
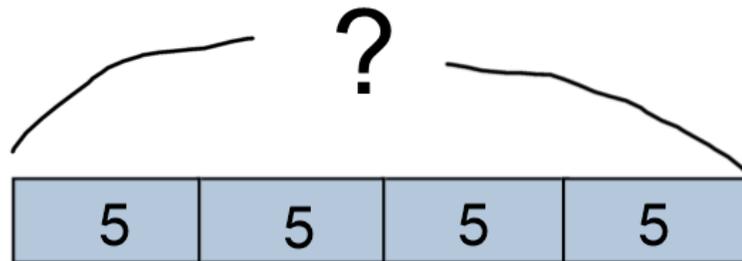
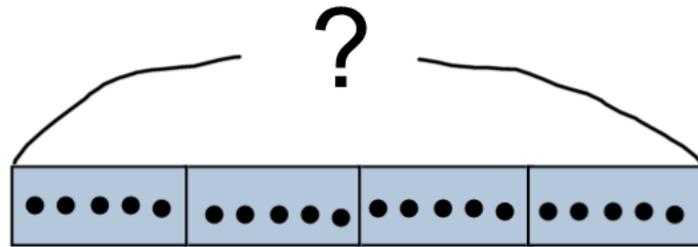


❖ Comparison

- $x$  is three times as much as  $y$

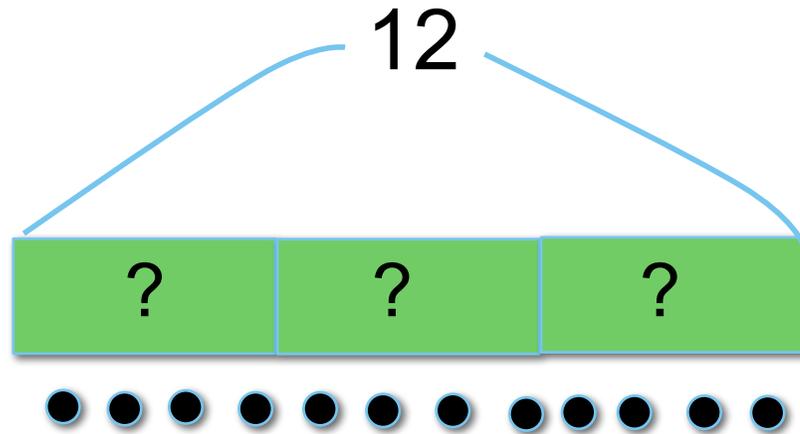


Robin has 4 packages of gum. There are 5 pieces in each package. How many pieces of gum does Robin have?



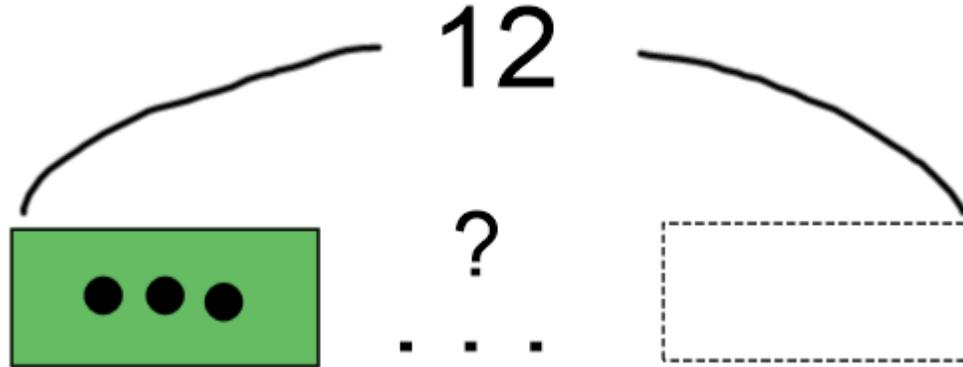
Structure	
Equal Groups	Multiplication <i>Problem Type</i>
Whole Unknown	

Mark has 12 cookies. He wants to put them on 3 plates so that he has the same number of cookies on each plate. How many cookies should he put on each plate?



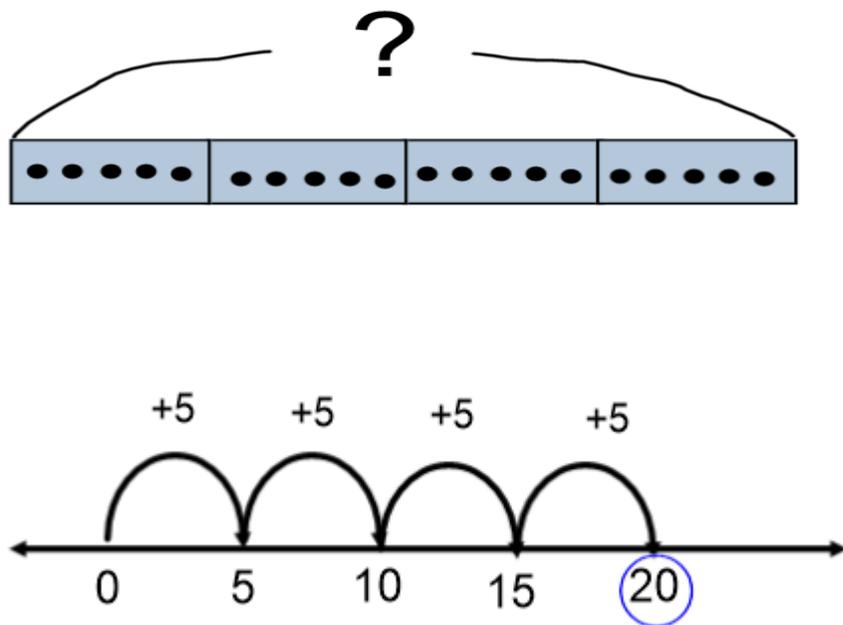
Structure	
Equal Groups	Partitive Division <i>Problem Type</i>
	<b>Size of Groups</b> Unknown

Kate has 12 cents to buy candy. If each gumdrop costs 3 cents, how many gumdrops can she buy?



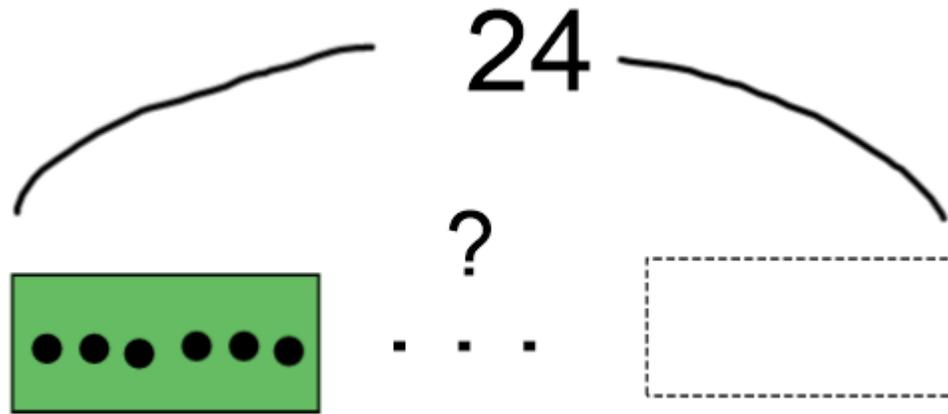
Structure	
Equal Groups	Measurement Division <i>Problem Type</i>
	<b>Number of Groups Unknown</b>

There were 4 cars. Each car had 5 people in it.  
How many people were riding in cars?



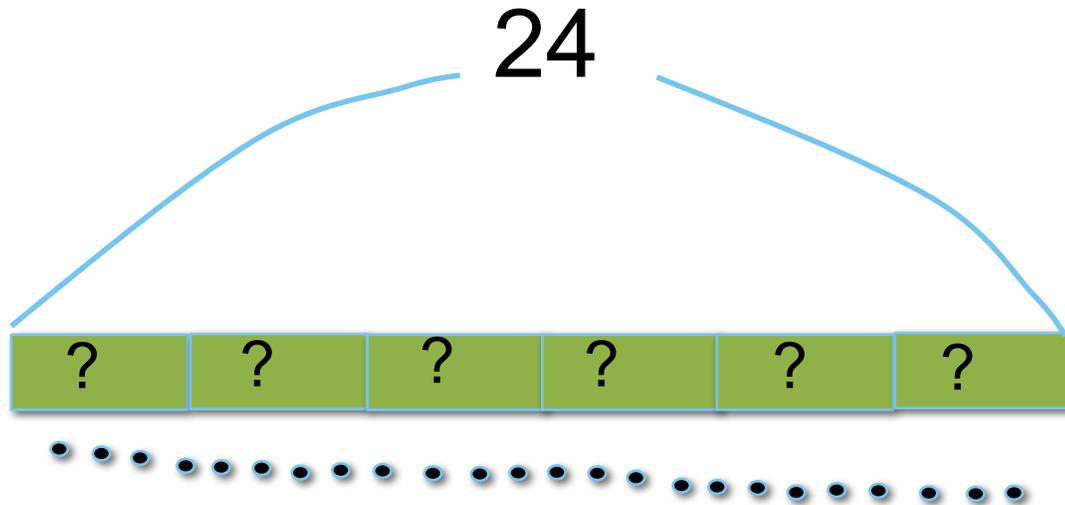
Structure	
Equal Groups	Multiplication <i>Problem Type</i>
Whole Unknown	

Mrs. Smith baked 24 fudge bars. She needs to pack them 6 to a box. How many boxes does Mrs. Smith need to pack all the fudge bars?



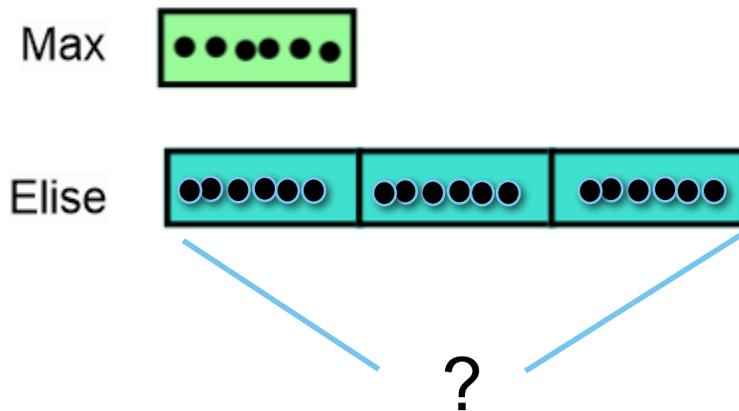
Structure	
Equal Groups	Measurement Division <i>Problem Type</i>
	<b>Number of Groups Unknown</b>

Mrs. Smith baked 24 fudge bars. She has 6 boxes. Each box must have the same amount of fudge bars. How many fudge bars need to go in each box?



Structure	
Equal Groups	Partitive Division <i>Problem Type</i>
	<b>Size of Groups Unknown</b>

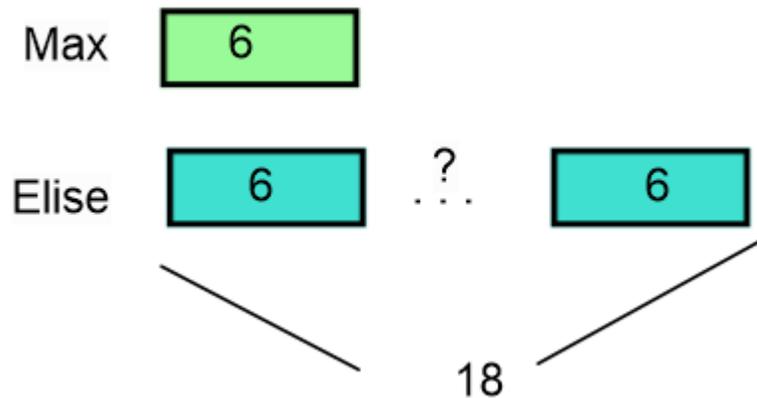
Max ate 6 cherries. Elise ate 3 times the amount of cherries than Max. How many cherries did Elise eat?



Structure	
Multiplicative Comparison	Product Unknown <i>Problem Type</i>
	Total Unknown

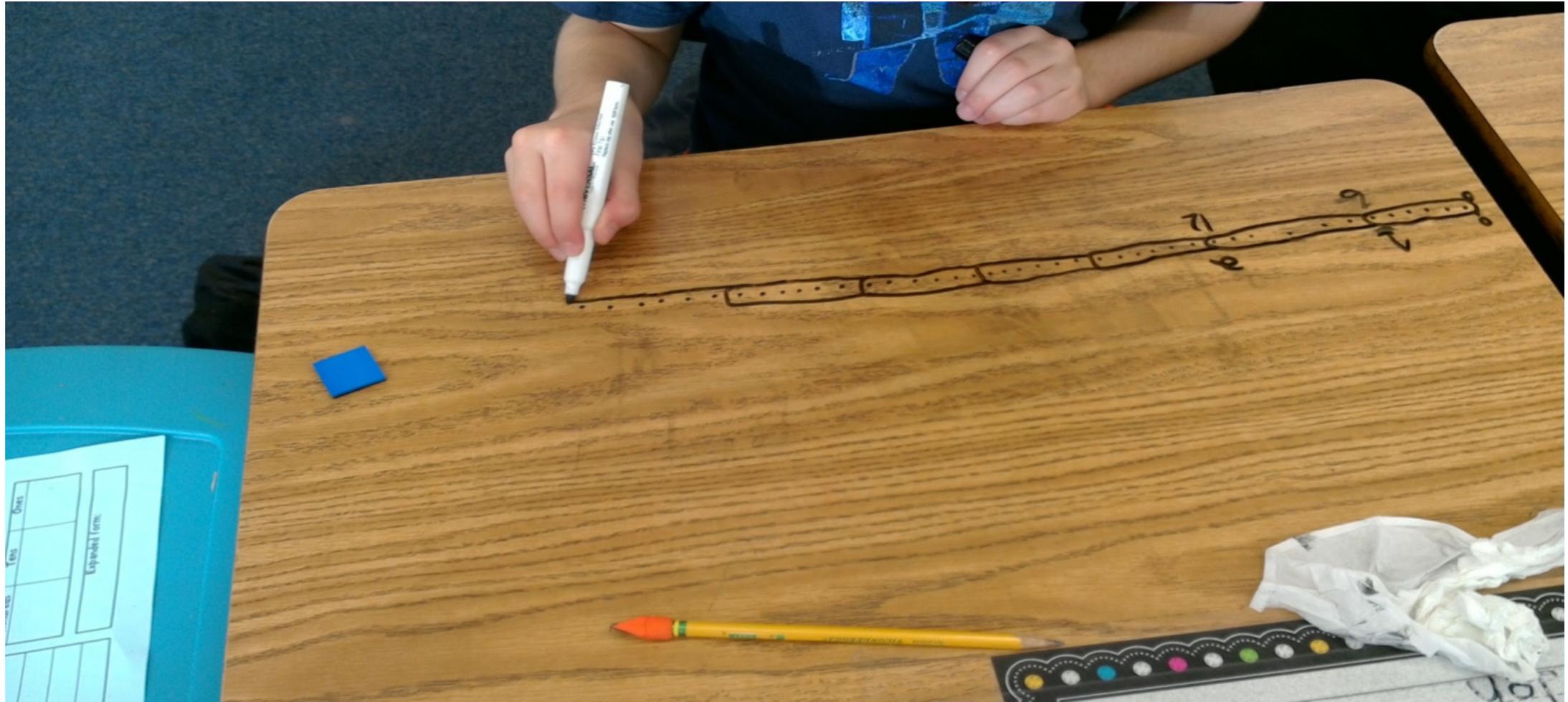


Elise ate 18 cherries. Max ate 6 cherries. How many times more cherries did Elise eat than Max?



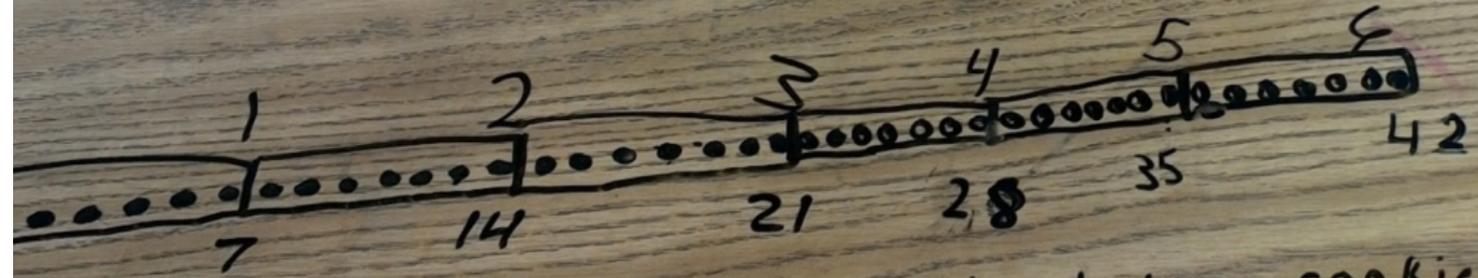
Structure	
Multiplicative Comparison	Measurement Division <i>Problem Type</i>
	<b>Number of Groups Unknown</b>

# Student work

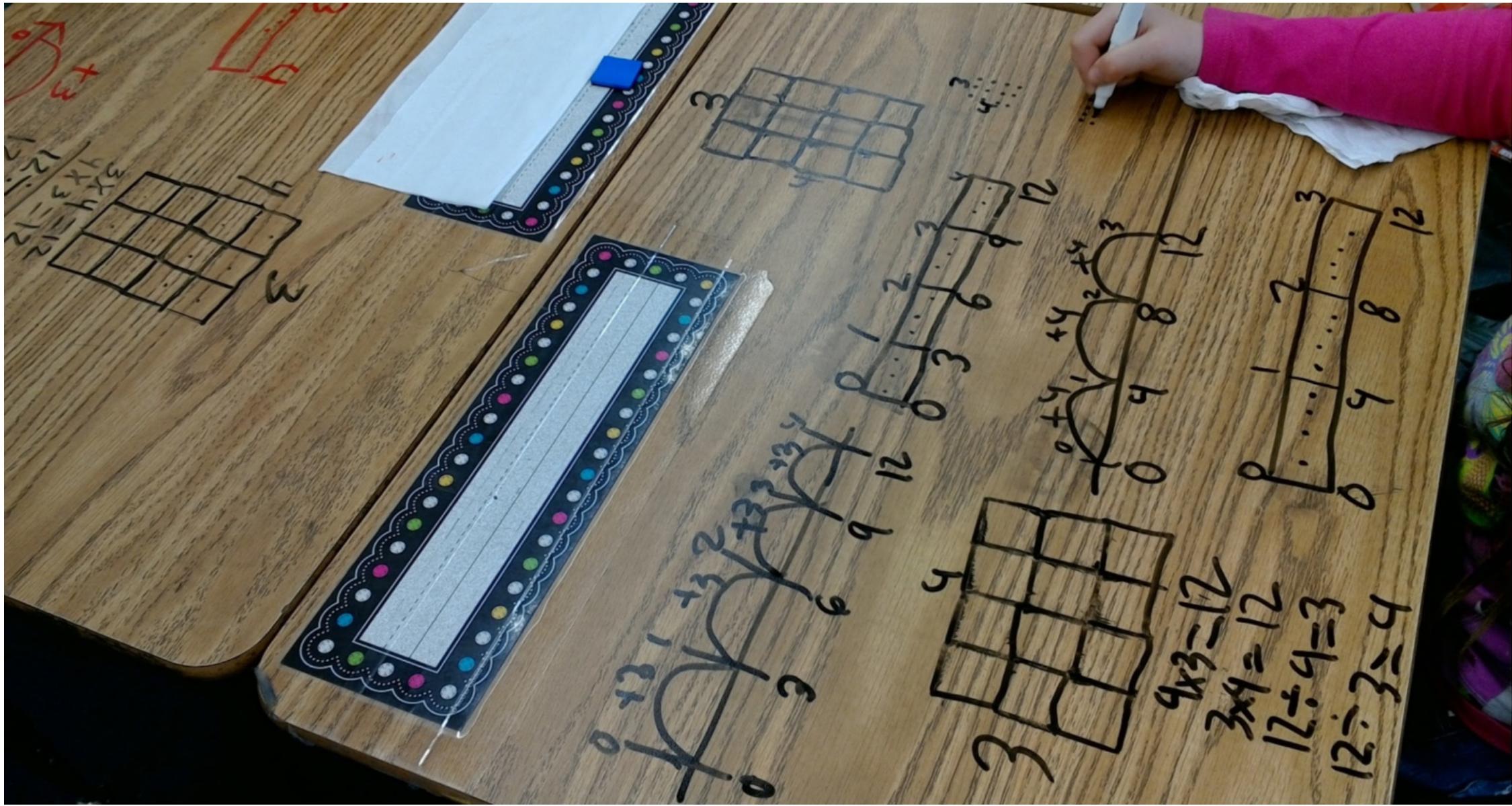


$$42 \div 6 = 7$$

$$42 \div 6 = 7$$



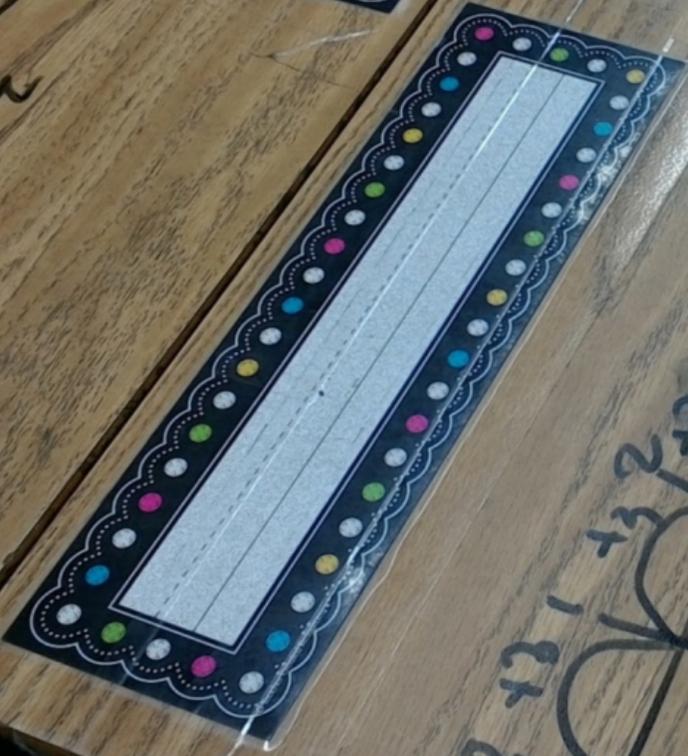
I had 42 cookies I put  
7 in each. I gave one box to  
each of my friends. How many friends  
got a box of cookies?



3x4=12  
12÷3=4  
12÷4=3

1	2	3	4
1	2	3	4
1	2	3	4

3



0 3 6 9 12

0	3	6	9	12
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3


3


$4 \times 3 = 12$   
 $3 \times 4 = 12$   
 $12 \div 4 = 3$   
 $12 \div 3 = 4$

0 3 6 9 12

0	3	6	9	12
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0 4 8 12

0	4	8	12
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0 3 6 9 12

0	3	6	9	12
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# NUMBERLESS WORD PROBLEMS

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Another Strategy to Illuminate Structure and Problem Type

Jack has toy cars. His mom gave him toy cars. How many toy cars does Jack have?

Bella read some pages at home. She read some more at school. Bella has now read some pages. How many pages did she read at school?

Owen had some crayons. Kendall gave him some more crayons. Now Owen has some crayons. How many crayons did Owen have to start with?

There were students in the cafeteria, but some students left. Now how many students are in the cafeteria?

Jay had some chips. He gave Caleb some chips. Now Jay has some potato chips. How many chips did Jay have before he gave some chips to Caleb?

Lisl had some marbles. She gave some to Thomas. Now she has some marbles left. How many marbles did Lisl give to Thomas?

Jack has some green toy cars and some blue toy cars. How many toy cars does Jack have?

Jack has some toy cars. He has some green toy cars and the rest are blue. How many blue toy cars does Jack have?

Justin's shoe is inches long. Annie's shoe is inches long. How much longer is Justin's shoe than Annie's shoe?

Carter has some cents. He has some more cents than Ellie. How much money does Ellie have?

Jon has some marbles. Kate has some more than Jon. How many marbles does Kate have?